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ScienceDirect
Indagationes Mathematicae 35 (2024) 516–522
indagationes mathematicae
www.elsevier.com/locate/indag

Rational points in translations of the Cantor set

Kan Jiang[a], Derong Kong[b], Wenxia Li[c], Zhiqiang Wang[c,*]

[a] *Department of Mathematics, Ningbo University, Ningbo, People's Republic of China*
[b] *College of Mathematics and Statistics, Center of Mathematics, Chongqing University, Chongqing 401331, People's Republic of China*
[c] *School of Mathematical Sciences, Key Laboratory of MEA (Ministry of Education) & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, People's Republic of China*

Received 21 October 2023; received in revised form 8 March 2024; accepted 28 March 2024

Communicated by K. Dajani

Abstract

Given two coprime integers $p \geq 2$ and $q \geq 3$, let $D_p \subset [0, 1)$ consist of all rational numbers which have a finite p-ary expansion, and let

$$K(q,\mathcal{A}) = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{q^i} : d_i \in \mathcal{A} \ \forall i \in \mathbb{N} \right\},$$

where $\mathcal{A} \subset \{0, 1, \ldots, q-1\}$ with cardinality $1 < \#\mathcal{A} < q$. In 2021 Schleischitz showed that $\#(D_p \cap K(q,\mathcal{A})) < +\infty$. In this paper we show that for any $r \in \mathbb{Q}$ and for any $\alpha \in \mathbb{R}$,

$$\#\big((rD_p + \alpha) \cap K(q,\mathcal{A})\big) < +\infty.$$

Keywords: Cantor set; Rational number; q-ary expansion

1. Introduction

Given $q \in \mathbb{N}_{\geq 3}$ and $\mathcal{A} \subset \{0, 1, \ldots, q-1\}$ with cardinality $1 < \#\mathcal{A} < q$, we define the Cantor set $K(q,\mathcal{A}) \subset [0, 1]$ by

$$K(q,\mathcal{A}) := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{q^i} : d_i \in \mathcal{A} \ \forall i \in \mathbb{N} \right\}.$$

* Corresponding author.
E-mail addresses: jiangkan@nbu.edu.cn (K. Jiang), derongkong@126.com (D. Kong), wxli@math.ecnu.edu.cn (W. Li), zhiqiangwzy@163.com (Z. Wang).

https://doi.org/10.1016/j.indag.2024.03.012
0019-3577/

In particular, the choice of $q = 3$ and $\mathcal{A} = \{0, 2\}$ corresponds to the classical middle-third Cantor set. Given $p \in \mathbb{N}_{\geq 2}$, we define

$$D_p := \left\{ \sum_{i=1}^{n} \frac{d_i}{p^i} : d_i \in \{0, 1, \ldots, p-1\}\ \forall 1 \leq i \leq n;\ n \in \mathbb{N} \right\}. \tag{1.1}$$

Then D_p consists of all rational numbers in $[0, 1)$ which have a finite p-ary expansion. It is clear that D_p is countably infinite and dense in $[0, 1]$.

When $p = 10, q = 3$ and $\mathcal{A} = \{0, 2\}$, Wall [7] showed that

$$D_{10} \cap K(3, \{0, 2\}) = \left\{ \frac{1}{4}, \frac{3}{4}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{40}, \frac{3}{40}, \frac{9}{40}, \frac{13}{40}, \frac{27}{40}, \frac{31}{40}, \frac{37}{40}, \frac{39}{40} \right\}.$$

Later, Nagy [4] proved that for each prime number $p \in \mathbb{N}_{\geq 4}$, the set $D_p \cap K(3, \{0, 2\})$ is finite. Bloshchitsyn [1] generalized this result and proved that if $p > q^2$ is a prime number, then the set $D_p \cap K(q, \mathcal{A})$ is finite. The general result was recently obtained by Schleischitz [5, Corollary 4.4] (some further extensions can be found in [3,6]).

Theorem 1.1 (*Schleischitz, 2021*)**.** *Let $p \in \mathbb{N}_{\geq 2}$ and $q \in \mathbb{N}_{\geq 3}$ with $\mathbf{gcd}(p, q) = 1$. If $\mathcal{A} \subset \{0, 1, \ldots, q-1\}$ with $1 < \#\mathcal{A} < q$, then we have*

$$\#\big(D_p \cap K(q, \mathcal{A})\big) < +\infty.$$

In this paper we extend Theorem 1.1 as follows.

Theorem 1.2. *Let $p \in \mathbb{N}_{\geq 2}$ and $q \in \mathbb{N}_{\geq 3}$ with $\mathbf{gcd}(p, q) = 1$. If $\mathcal{A} \subset \{0, 1, \ldots, q-1\}$ with $1 < \#\mathcal{A} < q$, then for any $r \in \mathbb{Q}$ and any $\alpha \in \mathbb{R}$ we have*

$$\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty.$$

Note that in Theorem 1.1 the intersection $D_p \cap K(q, \mathcal{A})$ contains only rational numbers, while in Theorem 1.2 the intersection $(rD_p + \alpha) \cap K(q, \mathcal{A})$ involves irrational numbers if $\alpha \notin \mathbb{Q}$. To prove Theorem 1.2, we may assume that $\#\mathcal{A} = q - 1$, which means the set $\mathcal{A}$ only misses one digit in $\{0, 1, \ldots, q-1\}$. In terms of Theorem 1.2 we make the following conjecture, which claims that the conclusion still holds also for irrational x.

Conjecture 1.3. *Under the same condition as in Theorem 1.2, the conclusion*

$$\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty$$

also holds for any $r \notin \mathbb{Q}$ and any $\alpha \in \mathbb{R}$.

We will prove our main result Theorem 1.2 in the next section.

2. Proof of Theorem 1.2

In the following we fix two coprimes $p \in \mathbb{N}_{\geq 2}, q \in \mathbb{N}_{\geq 3}$ and the digit set $\mathcal{A} \subset \{0, 1, \ldots, q-1\}$ with $\#A = q - 1$. For a real number $x \in \mathbb{R}$, we write $\langle x \rangle$ for the fractional part of x, i.e., $\langle x \rangle \in [0, 1)$ and $x - \langle x \rangle \in \mathbb{Z}$. For $x \in [0, 1)$, the q-ary expansion of x is the sequence (x_i) in $\{0, 1, \ldots, q-1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}.$$

The q-ary expansion is unique except for countably many points that have precisely two q-ary expansions, one is finite and the other one ends in a periodic sequence with period $q-1$. For convenience, for these countably many exceptional points, the q-ary expansion refers to the finite expansion. In order to prove Theorem 1.2 we need the following lemma which can be deduced from Theorem 1.1.

Lemma 2.1. *Let $d_1d_2\dots d_k \in \{0,1,\dots,q-1\}^k$ be a block and $r \in \mathbb{Q}\setminus\{0\}$. Then for all but finitely many $x \in rD_p$, the block $d_1d_2\dots d_k$ occurs in the q-ary expansion of $\langle x\rangle$ infinitely often.*

Proof. Write $r = s/t$ with $s \in \mathbb{Z}$, $t \in \mathbb{N}$, and $\mathbf{gcd}(s,t) = 1$. We can find $\ell \in \mathbb{N}$ such that

$$\mathbf{gcd}\Big(\frac{t}{\mathbf{gcd}(t,q^\ell)}, q\Big) = 1. \tag{2.1}$$

Let $h := t/\mathbf{gcd}(t,q^\ell)$. Since $\mathbf{gcd}(h,q) = \mathbf{gcd}(p,q) = 1$, we have $\mathbf{gcd}(hp,q^k) = 1$. By Theorem 1.1, we have

$$\#\big(D_{hp} \cap K(q^k,\mathcal{B})\big) < +\infty, \tag{2.2}$$

where

$$\mathcal{B} = \big\{0,1,\dots,q^k-1\big\} \setminus \big\{d_1q^{k-1} + d_2q^{k-2} + \cdots + d_{k-1}q + d_k\big\}.$$

This implies that for any $y \in D_{hp}\setminus K(q^k,\mathcal{B})$, the block $d_1d_2\dots d_k$ occurs in the q-ary expansion of y. Note that each $y \in D_{hp}$ has a purely periodic q-ary expansion because $\mathbf{gcd}(hp,q) = 1$ (cf. [2, Proposition 2.1.2]). Thus, for any $y \in D_{hp}\setminus K(q^k,\mathcal{B})$, the block $d_1d_2\dots d_k$ occurs in the q-ary expansion of y infinitely often.

For ℓ defined in (2.1), consider the function f defined by

$$f: \quad \mathbb{R} \to [0,1); \quad x \mapsto \langle q^\ell x\rangle.$$

Note that the q-ary expansions of $\langle x\rangle$ and $f(x)$ have the same tail. Then for each $x \in f^{-1}\big(D_{hp}\setminus K(q^k,\mathcal{B})\big)$, the block $d_1d_2\dots d_k$ occurs in the q-ary expansion of $\langle x\rangle$ infinitely often. It suffices to show that

$$\#\Big((rD_p)\setminus f^{-1}\big(D_{hp}\setminus K(q^k,\mathcal{B})\big)\Big) < +\infty. \tag{2.3}$$

By (1.1), we can rewrite D_p as

$$D_p = \bigcup_{n=1}^{\infty}\Big\{\frac{d}{p^n} : d \in \{0,1,\dots,p^n-1\}\Big\}.$$

Note that $h = t/\mathbf{gcd}(t,q^\ell)$. Then for any $d/p^n \in D_p$, we have

$$\begin{aligned} f\Big(r\cdot\frac{d}{p^n}\Big) &= \Big\langle q^\ell \cdot \frac{s}{t}\cdot\frac{d}{p^n}\Big\rangle \\ &= \Big\langle \frac{q^\ell}{\mathbf{gcd}(t,q^\ell)}\cdot\frac{s}{h}\cdot\frac{d}{p^n}\Big\rangle \\ &= \Big\langle \frac{q^\ell}{\mathbf{gcd}(t,q^\ell)}\cdot\frac{sh^{n-1}d}{(hp)^n}\Big\rangle \in D_{hp}. \end{aligned}$$

So, we obtain that $f(rD_p) \subset D_{hp}$, and then $rD_p \subset f^{-1}(D_{hp})$. This implies that

$$(rD_p)\setminus f^{-1}\big(D_{hp}\setminus K(q^k,\mathcal{B})\big) = \Big((rD_p)\cap f^{-1}(D_{hp})\Big)\setminus f^{-1}\big(D_{hp}\setminus K(q^k,\mathcal{B})\big)$$

$$= (rD_p) \cap \Big(f^{-1}(D_{hp}) \setminus f^{-1}\big(D_{hp} \setminus K(q^k, \mathcal{B})\big)\Big)$$
$$= (rD_p) \cap f^{-1}\big(D_{hp} \cap K(q^k, \mathcal{B})\big).$$

Next, we show that f is finite-to-one on rD_p, i.e., for any $y \in D_{hp}$, $\#\big((rD_p) \cap f^{-1}(\{y\})\big) < +\infty$. Then, by (2.2) we obtain (2.3) as desired.

Let $p_1, p_2, \ldots, p_u$ be all distinct prime factors of p. Then each point in $D_p \setminus \{0\}$ has the unique representation of the form

$$\frac{a}{p_1^{n_1} p_2^{n_2} \cdots p_u^{n_u}}$$

with $a \in \mathbb{N}$, $\mathbf{gcd}(a, p) = 1$ and $n_1, n_2, \ldots, n_u$ are all nonnegative integers. For any $y = d/(hp)^n \in D_{hp}$, if

$$f\Big(r \cdot \frac{a}{p_1^{n_1} p_2^{n_2} \cdots p_u^{n_u}}\Big) = \Big\langle q^\ell \cdot \frac{s}{t} \cdot \frac{a}{p_1^{n_1} p_2^{n_2} \cdots p_u^{n_u}}\Big\rangle$$
$$= \Big\langle \frac{q^\ell}{\mathbf{gcd}(t, q^\ell)} \cdot \frac{sa}{hp_1^{n_1} p_2^{n_2} \cdots p_u^{n_u}}\Big\rangle$$
$$= y = \frac{d}{(hp)^n},$$

then by using $\mathbf{gcd}(a, p) = 1$ and $\mathbf{gcd}(q, p) = 1$ we obtain that

$$p_1^{n_1} p_2^{n_2} \cdots p_u^{n_u} \mid s(hp)^n.$$

This implies that all possible $n_1, n_2, \ldots, n_u$ are bounded. Thus, we conclude that $\#\big((rD_p) \cap f^{-1}(\{y\})\big) < +\infty$, completing the proof. □

We also need the following lemma.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$ and $d \in \{1, 2, \ldots, q-2\}$ so that $q - d \notin \mathcal{A}$. If the q-ary expansion (α_i) of $\langle\alpha\rangle$ satisfies $\alpha_i \geq d$ for all $i \in \mathbb{N}$, then*

$$\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty \tag{2.4}$$

for any $r \in \mathbb{Q}$.

Proof. If $r = 0$, then (2.4) holds trivially. In the following we fix $r \in \mathbb{Q} \setminus \{0\}$. Take $\alpha \in \mathbb{R}$ such that the q-ary expansion (α_i) of $\langle\alpha\rangle$ satisfies $\alpha_i \in \{d, d+1, \ldots, q-1\}$ for all $i \in \mathbb{N}$. We will prove (2.4) according to different properties of (α_i). More precisely, for $0 \leq k \leq q - d$ let

$$A_k := \Big\{(a_i) \in \{d, d+1, \ldots, q-1\}^{\mathbb{N}} : \text{the length of any block in } (a_i) \text{ with each digit } < q-k \text{ is uniformly bounded}\Big\}.$$

Then $A_0 = \emptyset$ and $A_{q-d} = \{d, d+1, \ldots, q-1\}^{\mathbb{N}}$. Note that $A_k \subset A_{k+1}$ for $0 \leq k < q - d$. Thus,

$$(A_{q-d} \setminus A_{q-d-1}) \cup \cdots \cup (A_2 \setminus A_1) \cup (A_1 \setminus A_0) = A_{q-d} \setminus A_0 = \{d, d+1, \ldots, q-1\}^{\mathbb{N}}.$$

So it suffices to prove that for any $0 \leq k < q - d$,

$$\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty \quad \text{if} \quad (\alpha_i) \in A_{k+1} \setminus A_k. \tag{2.5}$$

We will split the proof of (2.5) into two cases: $0 \le k < d$, and $d \le k < q - d$ (assuming $2d < q$).

Case 1. $(\alpha_i) \in A_{k+1} \setminus A_k$ for some $0 \le k < d$. Since $(\alpha_i) \in A_{k+1}$, the set

$$\{n_1, n_2, n_3, \ldots\} := \{i \in \mathbb{N} : \alpha_i \ge q - k - 1\} \tag{2.6}$$

is infinite, where $n_1 < n_2 < n_3 < \cdots$, and there exists $m \in \mathbb{N}$ such that $n_{i+1} - n_i \le m$ for all $i \in \mathbb{N}$. Note that $0 \le k < d < q$. Then $1 \le q - d + k \le q - 1$.

Take $x \in rD_p$ such that the block $(q - d + k)^{m+1}0$ occurs in the q-ary expansion (x_i) of $\langle x \rangle$ infinitely often, and let

$$\{k_1, k_2, k_3, \ldots\} := \{i \in \mathbb{N}_{\ge n_1} : x_{i+1}x_{i+2}\cdots x_{i+m+2} = (q - d + k)^{m+1}0\}, \tag{2.7}$$

where $k_1 < k_2 < k_3 < \cdots$. Note that the q-ary expansion of $\langle x \rangle$ is eventually periodic. Then the sequence $\{k_{i+1} - k_i\}_{i\in\mathbb{N}}$ is bounded.

Claim. *there exists $i_0 \in \mathbb{N}$ such that $\alpha_{k_{i_0}+j} \le q - k - 1$ for all $1 \le j \le m + 2$.*

Suppose on the contrary that the claim fails. Then for each $i \in \mathbb{N}$ the block $\alpha_{k_i+1}\alpha_{k_i+2}\cdots\alpha_{k_i+m+2}$ contains at least one digit $\ge q - k$. Note that $\{k_{i+1} - k_i\}_{i\in\mathbb{N}}$ is bounded. Then the length of any block in (α_i) with each digit $< q - k$ is uniformly bounded. This implies that $(\alpha_i) \in A_k$, leading to a contradiction with our assumption.

Since the sequence $\{n_{i+1} - n_i\}_{i\in\mathbb{N}}$ is bounded by m and $k_{i_0} \ge n_1$, there exists $1 \le m' \le m$ such that $k_{i_0} + m' \in \{n_i\}$, and thus by (2.6) and the claim it follows that $\alpha_{k_{i_0}+m'} = q - k - 1$. Thus, by (2.7) and using $\alpha_i \ge d$ for all $i \in \mathbb{N}$ we obtain that

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{x_{k_{i_0}+m'-1+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+m'-1+i}}{q^i} &> \left(\frac{q-d+k}{q} + \frac{q-d+k}{q^2}\right) + \left(\frac{q-k-1}{q} + \frac{d}{q^2}\right) \\ &\ge 1 + \frac{q-d}{q},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{x_{k_{i_0}+m'-1+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+m'-1+i}}{q^i} &< \left(\frac{q-d+k}{q} + \frac{q-d+k+1}{q^2}\right) + \frac{q-k}{q} \\ &\le 1 + \frac{q+1-d}{q}.\end{aligned}$$

Whence,

$$\begin{aligned}\left\langle q^{k_{i_0}+m'-1}(x+\alpha)\right\rangle &= \left\langle \sum_{i=1}^{\infty} \frac{x_{k_{i_0}+m'-1+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+m'-1+i}}{q^i} \right\rangle \\ &\in \left(\frac{q-d}{q}, \frac{q+1-d}{q}\right).\end{aligned}$$

This implies that $x + \alpha \notin K(q, \mathcal{A})$. So, applying Lemma 2.1 for the block $(q - d + k)^{m+1}0$ involved in the q-ary expansion of x in (2.7), it follows that $\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty$.

Case 2. $(\alpha_i) \in A_{k+1} \setminus A_k$ for some $d \le k < q - d$ with the assumption $2d < q$.

Similar to Case 1, since $(\alpha_i) \in A_{k+1}$, the set

$$\{n_1, n_2, n_3, \ldots\} := \{i \in \mathbb{N} : \alpha_i \ge q - k - 1\}$$

is infinite, where $n_1 < n_2 < n_3 < \cdots$, and there exists $m \in \mathbb{N}$ such that $n_{i+1} - n_i \le m$ for all $i \in \mathbb{N}$. Note that $1 \le d \le k < q - d$. Then $1 \le k - d + 1 \le q - 1$. Take $x \in rD_p$ such that the block $(k-d+1)^m 0$ occurs in the q-ary expansion (x_i) of $\langle x \rangle$ infinitely often, and write

$$\{k_1, k_2, \ldots\} := \{i \in \mathbb{N}_{\ge n_1} : x_{i+1}x_{i+2}\cdots x_{i+m+1} = (k-d+1)^m 0\}, \tag{2.8}$$

where $k_1 < k_2 < \cdots$. Note that the q-ary expansion of $\langle x \rangle$ is eventually periodic. Then the sequence $\{k_{i+1} - k_i\}_{i \in \mathbb{N}}$ is bounded. Since $(\alpha_i) \notin A_k$, by the same argument as in Case 1 we can find $i_0 \in \mathbb{N}$ such that $\alpha_{k_{i_0}+j} \le q-k-1$ for all $1 \le j \le m+1$, and there exists $1 \le m' \le m$ such that $k_{i_0} + m' \in \{n_i\}$ and $\alpha_{k_{i_0}+m'} = q - k - 1$. Thus,

$$\sum_{i=1}^{\infty} \frac{x_{k_{i_0}+m'-1+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+m'-1+i}}{q^i} > \frac{k-d+1}{q} + \frac{q-k-1}{q} = \frac{q-d}{q},$$

and

$$\begin{aligned}
&\sum_{i=1}^{\infty} \frac{x_{k_{i_0}+m'-1+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+m'-1+i}}{q^i} \\
&< \left(\sum_{i=1}^{m-m'+1} \frac{k-d+1}{q^i} + \frac{1}{q^{m-m'+2}} \right) + \left(\sum_{i=1}^{m-m'+2} \frac{q-k-1}{q^i} + \frac{1}{q^{m-m'+2}} \right) \\
&= \sum_{i=1}^{m-m'+1} \frac{q-d}{q^i} + \frac{q-k+1}{q^{m-m'+2}} \\
&\le \frac{q+1-d}{q}.
\end{aligned}$$

Whence,

$$\left\langle q^{k_{i_0}+m'-1}(x+\alpha) \right\rangle \in \left(\frac{q-d}{q}, \frac{q+1-d}{q} \right),$$

which implies $x + \alpha \notin K(q, \mathcal{A})$. Therefore, applying Lemma 2.1 for the block $(k-d+1)^m 0$ involved in the q-ary expansion of x in (2.8), we deduce that $\#\big((rD_p + \alpha) \cap K(q, \mathcal{A})\big) < +\infty$. This together with Case 1 proves (2.5), completing the proof. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. If $r = 0$, then it holds trivially. In the following we fix $r \in \mathbb{Q} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Let (α_i) denote the q-ary expansion of $\langle \alpha \rangle$. Since $\#\mathcal{A} = q - 1$, there exists $1 \le d \le q$ such that $q - d \notin \mathcal{A}$. We split the proof into two cases according to (α_i).

Case 1. For any $m \in \mathbb{N}$ the sequence (α_i) contains the block 0^m.

Fix $x \in rD_p$ such that the block $(q-d)0$ occurs in the q-ary expansion (x_i) of $\langle x \rangle$ infinitely often, and write

$$\{k_1, k_2, \ldots\} := \{i \in \mathbb{N} : x_{i+1}x_{i+2} = (q-d)0\}, \tag{2.9}$$

where $k_1 < k_2 < \cdots$. Note that the q-ary expansion of $\langle x \rangle$ is eventually periodic. Then the sequence $\{k_{i+1} - k_i\}_{i \in \mathbb{N}}$ is bounded. Observe that (α_i) contains arbitrarily long consecutive zeros. There exists $i_0 \in \mathbb{N}$ such that $\alpha_{k_{i_0}+1}\alpha_{k_{i_0}+2} = 00$. Thus, by using $x_{k_{i_0}+1}x_{k_{i_0}+2} = (q-d)0$ it follows that

$$\frac{q-d}{q} < \sum_{i=1}^{\infty} \frac{x_{k_{i_0}+i}}{q^i} + \sum_{i=1}^{\infty} \frac{\alpha_{k_{i_0}+i}}{q^i} < \frac{q+1-d}{q},$$

which implies $\langle q^{k_{i_0}}(x+\alpha)\rangle \notin K(q,\mathcal{A})$. So, $x+\alpha \notin K(q,\mathcal{A})$. Applying Lemma 2.1 for the block $(q-d)0$ involved in the q-ary expansion of x in (2.9), it follows that $\#\big((rD_p+\alpha)\cap K(q,\mathcal{A})\big) < +\infty$.

Case 2. There exists $m \in \mathbb{N}$ such that the sequence (α_i) does not contain the block 0^m. Note that $K(q,\mathcal{A}) = K(q',\mathcal{A}')$, where

$$q' = q^{m+1} \quad \text{and} \quad \mathcal{A}' = \left\{\sum_{i=0}^{m} \varepsilon_i q^i : \varepsilon_i \in \mathcal{A}\ \forall 0 \le i \le m\right\}.$$

Then it suffices to show that $\#\big((rD_p+\alpha)\cap K(q',\mathcal{A}')\big) < +\infty$. Since $\mathbf{gcd}(p,q)=1$, we have $\mathbf{gcd}(p,q')=1$. Observe that

$$\langle\alpha\rangle = \sum_{i=1}^{\infty}\frac{\alpha_i}{q^i} = \sum_{k=1}^{\infty}\frac{1}{q^{k(m+1)}}\left(\sum_{i=0}^{m}\alpha_{k(m+1)-i}q^i\right) = \sum_{k=1}^{\infty}\frac{\alpha'_k}{(q')^k},$$

where

$$\alpha'_k := \sum_{i=0}^{m}\alpha_{k(m+1)-i}q^i. \tag{2.10}$$

Then the sequence (α'_k) is the q'-ary expansion of $\langle\alpha\rangle$. Since the sequence (α_i) does not contain the block 0^m, for each $k\in\mathbb{N}$ there exists $1\le i_k\le m$ such that $\alpha_{k(m+1)-i_k} > 0$. It follows from (2.10) that $\alpha'_k \ge q \ge d$ for all $k\in\mathbb{N}$. Furthermore, since $q-d\notin\mathcal{A}$, we have

$$q'-d = q-d+(q-1)q+(q-1)q^2+\cdots+(q-1)q^m \notin \mathcal{A}'.$$

Clearly, we have $1\le d\le q < q'-1$. Thus, applying Lemma 2.2 for the set $K(q',\mathcal{A}')$ we conclude that $\#\big((rD_p+\alpha)\cap K(q',\mathcal{A}')\big) < +\infty$. □

Acknowledgments

The authors thank the anonymous referees for many useful suggestions which improve the presentation of the paper. K. Jiang was supported by NSFC No. 11701302, Zhejiang Provincial NSF No. LY20A010009, and the K.C. Wong Magna Fund in Ningbo University, China. D. Kong was supported by NSFC No. 11971079. W. Li was supported by NSFC No. 12071148 and Science and Technology Commission of Shanghai Municipality (STCSM) No. 22DZ2229014. Z. Wang was supported by the Fundamental Research Funds for the Central Universities, China No. YBNLTS2023-016.

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